METHODS FOR DETERMINING THERMOPHYSICAL CHARACTERISTICS OF SOLID MATERIALS BASED ON SOLUTIONS OF LINEAR HEAT-CONDUCTION EQUATIONS OF ORDER HIGHER THAN THE SECOND

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On the basis of solutions of the linearized heat-conduction equations, formulas are derived for the highly accurate determination of the thermophysical characteristics of solid materials.

Most methods of determining the temperature dependence of the thermophysical characteristics of solid materials are based on solutions of the linear heat-conduction equations. This leads to simpler computational formulas, but decreases the accuracy of the determination of the required quantities, and increases the cost and duration of an experiment.

We discuss methods based on solutions of the linearized heat-conduction equations which differ from the nonlinear equations

$$\rho_0 C(t) \frac{\partial t}{\partial \tau} = \frac{1}{r^k} \frac{\partial}{\partial r} \left[r^k \lambda(t) \frac{\partial t}{\partial r} \right] \quad (k = 0,\ 1,\ 2) \tag{1}$$

in the simplicity of the solution, a high degree of accuracy, and the possibility of constructing simple calculational formulas for the thermophysical characteristics of solid materials. One of the groups of linearized heat-conduction equations is given in [1] and has the form

$$\frac{\partial^{n+1} t}{\partial r^n \partial \tau} = a_0 \frac{\partial^n}{\partial r^n} \left[\frac{1}{r^k} \frac{\partial t}{\partial r} \left(r^k \frac{\partial t}{\partial r} \right) \right] \quad (k = 0,\ 1,\ 2), \tag{2}$$

where a_0 is the thermal diffusivity of the material at the initial temperature t_0. It is shown in [1] that for n = 1 (and more so for n = 2) the solutions of Eqs. (1) and (2) are rather close to one another.

A second group of linearized equations has the form [2]:

$$\frac{\partial^2 \Theta_1}{\partial \tau^2} = a_0 \frac{\partial^2}{\partial r^2} \left[\frac{1}{r^k} \frac{\partial \Theta_1}{\partial r} \left(r^k \frac{\partial \Theta_1}{\partial r} \right) \right] \quad (k = 0,\ 1,\ 2), \tag{3}$$

$$\frac{\partial^2 \Theta_2}{\partial \tau^2} = a_0^2 \frac{\partial^2}{\partial r^2} \left[\frac{1}{r^k} \frac{\partial \Theta_2}{\partial r} \left(r^k \frac{\partial \Theta_2}{\partial r} \right) \right] \quad (k = 0,\ 1,\ 2), \tag{4}$$

where

$$\Theta_1(r,\ \tau) = \frac{1}{\lambda_0} \int_{t_0}^{t} \lambda(t)\, dt; \tag{5}$$

$$\Theta_2(r,\ \tau) = \frac{1}{C_0} \int_{t_0}^{t} C(t)\, dt; \tag{6}$$

the thermal conductivity λ_0 and the specific heat C_0 are also evaluated at the initial temperature t_0.

In order to derive the required formulas, we consider an infinite plate of thickness 2R which is heated symmetrically and at zero time has the temperature t_0. The following boundary conditions are assumed:

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$$\Theta(x,\ \tau)|_{\tau=0} = 0 \quad (0 \leqslant x \leqslant R), \tag{7}$$

$$\left.\frac{\partial\Theta(x,\ \tau)}{\partial\tau}\right|_{\tau=0} = 0 \quad (0 \leqslant x \leqslant R), \tag{8}$$

$$\Theta(x,\ \tau)|_{x=0} = \varphi_0(\tau) \quad (\tau > 0), \tag{9}$$

$$\Theta(x,\ \tau)|_{x=R/2} = \varphi_{R/2}(\tau) \quad (\tau > 0), \tag{10}$$

$$\Theta(x,\ \tau)|_{x=R} = \varphi_R(\tau) \quad (\tau > 0), \tag{11}$$

$$\left.\frac{\partial\Theta(x,\ \tau)}{\partial x}\right|_{x=0} = 0 \quad (\tau \geqslant 0), \tag{12}$$

$$\lambda(\Theta)\left.\frac{\partial\Theta(x,\ \tau)}{\partial x}\right|_{x=R} = q(\tau) \quad (\tau > 0), \tag{13}$$

where $\Theta(x,\ \tau) = t(x,\ \tau) - t_0$.

Let us consider first Eq. (2) for n = 1 and k = 0, i.e.,

$$\frac{\partial^2\Theta}{\partial x\partial\tau} = a_0\frac{\partial^3\Theta}{\partial x^3}. \tag{14}$$

Taking the Laplace transform of Eq. (14) and using (7), (9), (11), and (12), we obtain

$$T(x,\ s) = \varphi_0(s) + [\varphi_R(s) - \varphi_0(s)]\frac{\operatorname{ch}\sqrt{\frac{s}{a_0}}\,x - 1}{\operatorname{ch}\sqrt{\frac{s}{a_0}}\,R - 1}. \tag{15}$$

Taking the inverse transform and limiting ourselves to first-order derivatives of $\varphi_0(\tau)$ and $_R(\tau)$, we find

$$\Theta(x,\ \tau) = \varphi_0(\tau) + [\varphi_R(\tau) - \varphi_0(\tau)]\cdot\frac{x^2}{R^2} + [\varphi_R'(\tau) - \varphi_0'(\tau)]\,\frac{x^2(x^2 - R^2)}{12a_0R^2}. \tag{16}$$

The restriction to first-order derivatives of $\varphi_0(\tau)$ and $\varphi_R(\tau)$, as shown in practice [3,4], is fully justified, since in the final solution the terms containing second derivatives are negligibly small in comparison with those containing first derivatives.

An analysis of Eq. (1) for k = 0 and $x \to 0$ shows that

$$\rho_0 C(t)\left.\frac{\partial t}{\partial\tau}\right|_{x=0} = \lambda(t)\left.\frac{\partial^2 t}{\partial x^2}\right|_{x=0} \tag{17}$$

or

$$a(t)|_{x=0}\left.\left[\frac{\partial t}{\partial\tau}\left(\frac{\partial^2 t}{\partial x^2}\right)^{-1}\right]\right|_{x=0}. \tag{18}$$

Taking account of previous remarks and Eq. (16), we obtain the required expression for the thermal diffusivity in the form

$$a(t) = \frac{\varphi_0'(\tau)\,R^2}{2(\varphi_R(\tau) - \varphi_0(\tau)) - (\varphi_R'(\tau) - \varphi_0'(\tau))\,\frac{1}{6a_0}}. \tag{19}$$

If the plate is heated at the rate b under quasisteady conditions, Eq. (19) leads directly to the well-known formula [5]

$$a = \frac{bR^2}{2\Delta t}, \tag{20}$$

which is fundamentally different from (19). This difference consists first in the fact that Eq. (20) was derived from the solution of the linear second-order heat-conduction equation and can be applied for temperature drops of the order of 10-15°C between the boundary surfaces. Second, when it is used there remains open the question of the temperature (at the center, on the surface, or their average) to which the values obtained refer. Equation (19) is obtained from the solution of the linear third-order heat-conduction equation, and is therefore more accurate. In addition, it lacks that indeterminancy which was mentioned in connection with Eq. (20), since, according to (19), the thermal diffusivity is determined at the point x = 0. Finally, it does not require quasisteady heating conditions and does not limit the temperature drop between the boundary surfaces. Further, let us consider Eq. (2) for n = 2 and k = 0. Taking account of (7) and (9)-(11), the solution of its transform is

$$T(x,\ s)=\varphi_0(s)+[\varphi_R(s)-\varphi_0(s)]\ \frac{\delta(l,\ x,\ s)}{\delta(l,\ R,\ s)}+[\varphi_l(s)-\varphi_0(s)]\ \frac{\delta(R,\ x,\ s)}{\delta(R,\ l,\ s)}, \tag{21}$$

where

$$\delta(l,\ x,\ s)=\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ l-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ x-\sqrt{\frac{s}{a_0}}\ x\right)-\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ x-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ l-\sqrt{\frac{s}{a_0}}\ l\right);$$

$$\delta(R,\ x,\ s)=\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ x-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ R-\sqrt{\frac{s}{a_0}}\ R\right)-\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ R-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ x-\sqrt{\frac{s}{a_0}}\ x\right);$$

$$\delta(R,\ l,\ s)=\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ l-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ R-\sqrt{\frac{s}{a_0}}\ R\right)-\left(\mathrm{ch}\sqrt{\frac{s}{a_0}}\ R-1\right)\left(\mathrm{sh}\sqrt{\frac{s}{a_0}}\ l-\sqrt{\frac{s}{a_0}}\ l\right).$$

The complexity of the denominator makes finding the inverse transform rather complicated. However, by using the procedure given above, we find that

$$\Theta(x,\ \tau)=\varphi_0(\tau)+(\varphi_R(\tau)-\varphi_0(\tau))\ \frac{x^2(x-l)}{R^2(R-l)}+$$

$$+\ (\varphi_2'(\tau)-\varphi_0'(\tau))\ \Delta_1(R,\ l,\ x)+(\varphi_l(\tau)-\varphi_0(\tau))\frac{x^2(R-x)}{l^2(R-l)}\ \frac{x^2(R-x)}{l^2(R-l)}+(\varphi_l'(\tau)-\varphi_0'(\tau))\ \Delta_2(R,\ l,\ x), \tag{22}$$

where

$$\Delta_1(R,\ l,\ x)=\frac{x^2(x-l)(x-R)(3x+3l-2R)}{60a_0R^2(R-l)};$$

$$\Delta_2(R,\ l,\ x)=\frac{x^2(l-x)(R-x)(2R-3l-3x)}{60a_0l^2(R-l)}.$$

As before, we have from (18)

$$a(t)|_{x=0}=\frac{\varphi_0'(\tau)}{F(\tau)+\varphi(\tau)}, \tag{23}$$

where

$$F(\tau)=[\varphi_l(\tau)-\varphi_0(\tau)]\ \frac{2R}{l^2(R-l)}-[\varphi_R(\tau)-\varphi_0(\tau)]\ \frac{2l}{R^2(R-l)};$$

$$\Phi(\tau)=[\varphi_l'(\tau)-\varphi_0'(\tau)]\frac{R(2R-3l)}{30a_0l(R-l)}-[\varphi_R'(\tau)-\varphi_0'(\tau)]\ \frac{R(2l-3R)}{30a_0l(R-l)}.$$

Equation (23) appears somewhat more complicated than (19). It is more accurate, however, since it was obtained from the solution of the linear fourth-order heat-conduction equation. It is not difficult to determine the thermal conductivity also. Thus, e.g., if during the experiment the heat flux q(τ) is given, then using (16), (22), and (13) we obtain expressions for the thermal conductivity of the material under study.

It is of interest to consider the use of the linearized heat-conduction equations of the second group to determine the thermophysical characteristics of materials. Let us assume that the thermal conductivity of the material under study is known beforehand. In this case it is expedient to use Eq. (3). Thus, for symmetric heating of an infinite plate it has the form

$$\frac{\partial^2\Theta_1}{\partial\tau^2}=a_0^2\ \frac{\partial^4\Theta_1}{\partial x^4}, \tag{24}$$

and the following boundary conditions can be specified:

$$\Theta_1(x,\ \tau)|_{\tau=0}=0\quad(0\leqslant x\leqslant R), \tag{25}$$

$$\left.\frac{\partial\Theta_1(x,\ \tau)}{\partial\tau}\right|_{\tau=0}=0\quad(0\leqslant x\leqslant R), \tag{26}$$

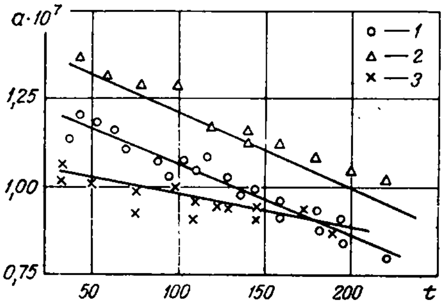

Fig. 1. Temperature dependence of the thermal diffusivity of polytetrafluoroethylene: 1) by Eq. (19); 2) from data in [6]; 3) by Eq. (23). $a \cdot 10^7$ is in m^2/sec, and t is in °C.

$$\Theta_1(x,\ \tau)|_{x=0} = \Phi_0(\tau) \quad (\tau > 0), \tag{27}$$

$$\Theta_1(x,\ \tau)|_{x=R/2} = \Phi_{R/2}(\tau) \quad (\tau > 0), \tag{28}$$

$$\Theta_1(x,\ \tau)|_{x=R} = \Phi_R(\tau) \quad (\tau > 0), \tag{29}$$

$$\left.\frac{\partial\Theta_1(x,\ \tau)}{\partial x}\right|_{x=0} = 0 \quad (\tau \geqslant 0), \tag{30}$$

where the functions $\Phi_0(\tau)$, $\Phi_{R/2}(\tau)$, and $\Phi_R(\tau)$ are found from Eq. (5). Applying the procedure described above, we write down the solution of the system given:

$$\begin{aligned}\Theta_1(x,\ \tau) = \Phi_0(\tau)\left[\frac{R(l^2-x^2)}{l^2(R-l)} - \frac{l(R^2-x^2)}{R^2(R-l)} + \frac{x^3(x^2-l^2)}{R^2l^2(R-l)}\right] + \\ + \Phi_0'(\tau)\left[\frac{R(l^2-x^2)}{l^2(R-l)}(\delta_1(R,\ l,\ x) - \delta(R,\ l)) - \frac{l(R^2-x^2)}{R^2(R-l)}(\delta_2(R,\ l,\ x) - \right. \\ \left. -\delta(R,\ l)) + \frac{x^3(R^2-l^2)}{R^2l^2(R-l)}(\delta_3(R,\ l,\ x) - \delta(R,\ l))\right] + \Phi_l(\tau)\frac{x^2(R-x)}{l^2(R-l)} + \\ + \Phi'(\tau)\frac{x^2(R-x)}{l^2(R-l)}(\delta_4(R,\ x) - \delta(R,\ l)) + \Phi_R(\tau)\frac{x^2(x-l)}{R^2(R-l)} + \Phi_R'(\tau)\frac{x^2(x-l)}{R^2(R-l)}(\delta_5(l,\ x) - \delta(R,\ l)),\end{aligned} \tag{31}$$

where

$$\delta_1(R,\ l,\ x) = \frac{R^4}{840a_0^2} + \frac{l^4 + l^2x^2 + x^4}{360a_0^2} - \frac{l^2x^2}{24a_0^2};$$

$$\delta_2(R,\ l,\ x) = \frac{l^4}{840a_0^2} + \frac{R^4 + R^2x^2 + x^4}{360a_0^2} - \frac{R^2x^2}{24a_0^2};$$

$$\delta_3(R,\ l,\ x) = \frac{x^4}{840a_0^2} + \frac{R^4 + R^2l^2 + l^4}{360a_0^2} - \frac{R^2l^2}{24a_0^2};$$

$$\delta_4(R,\ x) = \frac{R^4 + R^3x + R^2x^2 + Rx^3 + x^4}{840a_0^2} - \frac{Rx(R^2 + Rx + x^2)}{360a_0^2};$$

$$\delta_5(l,\ x) = \frac{x^4 + x^3l + x^2l^2 + xl^3 + l^4}{840a_0^2} - \frac{xl(x^2 + xl + l^2)}{360a_0^2};$$

$$\delta_6(R,\ l) = \frac{R^4 + R^3l + R^2l^2 + Rl^3 + l^4}{840a_0^2} - \frac{Rl(R^2 + Rl + l^2)}{360a_0^2}.$$

By analyzing Eq. (1) for k = 0 and taking account of (5), we obtain the expression for the specific heat

$$C(\Theta)|_{x=0} = \frac{\lambda(t)}{\rho_0}\left.\left[\frac{\partial^2\Theta_1}{\partial x^2}(\Phi_0'(\tau))^{-1}\right]\right|_{x=0}. \tag{32}$$

Similarly, from the solution of Eq. (4) for k = 0 with boundary conditions similar to (25)-(30), an expression can be obtained for the thermal conductivity if the specific heat of the material is known beforehand.

The above procedure can easily be carried over to cylindrical and spherical symmetries. In addition, the methods presented are general, since they can be used under very different conditions of heating.

Figure 1 shows the results of experimental studies of the thermal diffusivity of polytetrafluoroethylene. It is clear from the figure that the higher the order of the equation the lower the position of the curve. Since Eq. (23) is based on the solution of the fourth-order linear heat-conduction equation, one should expect that it most completely reflects the true character of the variation of the temperature dependence of the thermal diffusivity of polytetrafluoroethylene.

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